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Theoretical refinements and practical implications of $\theta\beta$-ideal rough approximations

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Abstract. This study advances both the theoretical and practical aspects of $\theta\beta$-ideal approximation spaces by refining definitions, introducing new results, and proposing a comprehensive algorithm for real-world applications. By systematically addressing inaccuracies in previous formulations, we reinforce the foundational principles of $\theta\beta$-ideal rough set models, ensuring greater consistency and scientific rigor while offering essential complementary results. Furthermore, our approach generalizes existing methodologies, including those developed by Abd El-Monsef et al. (2014) and M. Hosny (2020), thereby enhancing both accuracy and applicability. The proposed mathematical algorithm offers a robust computational framework adaptable to various domains, such as medicine, chemistry, and economics, facilitating improved decision-making processes. These contributions establish a stronger theoretical foundation while expanding the practical relevance of $\theta\beta$-ideal approximation spaces.

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1. Introduction

The philosophy of rough sets [1] addresses challenges related to uncertainty, with its initial foundations relying on equivalence relations. However, this restriction limited the broader applicability of the approach. To overcome this limitation, rough set models were extended to incorporate general binary relations instead of equivalence relations, enabling significant advancements in the field [2, 3, 4, 5, 6] and the development of rough neighborhood-based approaches [7, 8, 9, 10].

Topology and its extensions have also been utilized in the study and generalization of this theory, as seen in [11, 12, 13, 14, 15], and in presenting various real-life applications across multiple fields [7, 8, 16, 17]. Topological structures encompass a rich array of concepts and results, offering applications in diverse areas, as discussed in [18, 19, 20, 21, 22]. These structures have been applied within various theoretical frameworks, including covering-based soft rough sets [11, 23, 24], fuzzy set theory [25, 26, 27, 28], and extended rough set theories [29, 30, 31, 32].

On the other hand, in recent years many papers and proposals have been published in various fields, such as cluster soft sets and cluster soft topologies [33], r-fuzzy soft δ-open sets [34], and soft topological spaces with applications [35, 36]. Additionally, fuzzy sets have been studied in [37, 38, 39]. Additionally, topology and its extensions have been widely applied in different contexts, as evidenced by studies [16, 40, 41, 42].

A notable contribution came in 2007 when Abo Khadra et al. [43] introduced a topological framework based on the concepts of right and left neighborhoods [9]. Their approach provided a direct method for constructing a topology from binary relations, bypassing the need for a base or subbase. This innovative perspective significantly expanded the applicability of rough set theory within a topological framework. This methodology laid the groundwork for El-Bably's Master's thesis [15], which further investigated and developed the concept of near-open sets in rough set theory by introducing additional topological tools to enhance its practical applications. In this framework, a topology τ on a universe X is defined as:

$$\tau = \{A \subseteq X : \forall p \in A,\ N(p) \subseteq A\},$$

where $N(p)$ denotes the neighborhood of an element p in X. This approach has been employed in various studies to extend topological structures within rough set theory, thereby enriching the field with new tools and methodologies.

In 2014, Abd El-Monsef et al. [44] expanded upon the methodology of Abo Khadra et al. by introducing the concept of a ***j*-Neighborhood Space** (j-**NS**). This innovation advanced neighborhood-based rough set theory by incorporating new types of neighborhoods formed through intersections and unions of right and left neighborhoods, as initially described by Yao [9] (who constructed them from after and fore sets [45]), as well as minimal neighborhoods introduced by Allam et al. [7, 46]. These developments led to the establishment of eight distinct neighborhood types, offering a versatile framework for generalizing Pawlak's rough set model without imposing restrictive relational assumptions. The j-**NS** framework has since influenced numerous studies, broadening its application to diverse topological structures within rough set theory.

In 2021, El-Sayed et al. [47] introduced the concept of **initial-neighborhoods**, derived from right neighborhoods and defined as:

$$N_r^{\mathrm{i}}(p) = \{q \in X : N_r(p) \subseteq N_r(q)\},$$

where $N_r(p)$ denotes the right neighborhood of $p \in X$. They demonstrated the utility of initial neighborhoods in rough set theory and their capacity to address real-world challenges, such as the COVID-19 pandemic, through generalized nano-topology. Concurrently, Abu-Gdairi et al. [48] introduced the dual concept of **basic-neighborhoods**, defined as:

$$N_r^{\mathfrak{b}}(p) = \{q \in X : N_r(q) \subseteq N_r(p)\}.$$

The relationship between these neighborhoods was clarified in [44] as:

$$N_r^{\mathfrak{i}}(p) \cap N_r^{\mathfrak{b}}(p) = N_r^{\mathfrak{c}}(p),$$

where $N_r^{\mathfrak{c}}(p)$ represents the **core neighborhood** of p, as proposed in [41] and extended in [40, 42].

El-Gayar et al. [49] and Taher et al. [50, 51] conducted an in-depth analysis of the relationships between topologies generated by these neighborhoods and their corresponding approximations. Their work highlighted the significant role of these approximations in decision-making contexts, particularly in medical applications. Recently, El-Bably et al. [52] extended the concept of initial-neighborhoods by introducing the notion of **maximal neighborhoods** [53]. Building on this foundation, they developed new approximations for rough sets. Their methodology involved an innovative model consisting of 12 distinct types of approximations, which was successfully applied to a notable medical application for diagnosing COVID-19 variants using a topology-based framework. Additionally, R. A. Hosny et al. [54] introduced a new model based on the topological concept of "Primal approximation spaces." It is worth noting that both studies fundamentally rely on the concept of j-**NS**.

In 2022, Nawar et al. [55] introduced, for the first time, the concept of $\theta\beta$**-ideal approximation spaces**, marking a significant contribution. Their work [55] presented $\theta\beta$-open sets in rough set theory and their applications. It is important to note that the methodologies they proposed differ entirely from any similar concepts introduced in subsequent papers bearing the same name. They employed the topological notion of $\theta\beta$-open sets within the j-**NS** (for each $j \in J = \{r, \ell, \langle r\rangle, \langle \ell\rangle, u, i, \langle u\rangle, \langle i\rangle\}$) framework to generalize the work of Abd El-Monsef et al. [44]. Furthermore, they proposed the concept of $\theta\beta_j$-ideal approximation spaces, extending the approaches of both Abd El-Monsef et al. [44] and M. Hosny [56], with subsequent refinements in [57]. To illustrate the accuracy and practical relevance of $\theta\beta$-ideal rough sets, Nawar et al. presented a chemistry application. Their findings demonstrated greater precision and effectiveness compared to existing methods, thereby showcasing the potential of their approach.

The original paper [55] presented two approaches for generalization:

(1) **First Approach:**
The first approach introduced the concept of $\theta\beta_j$-approximation spaces as a generalization of the methods presented by Abd El-Monsef et al. (2014) [44]. Specifically, the topologies generated from j-**NS** were generalized by

applying the class of $\theta\beta_j$-open sets to the topologies τ_j using the closure (cl_j) and interior (int_j) operators of these topologies.

(2) **Second Approach:**
The second approach involved using $\theta\beta_j$-open sets via ideals $\mathcal{I}$ to generate $\mathcal{I}$-$\theta\beta_j$-approximation spaces. This method generalizes the topologies $\tau_j^{\mathcal{I}}$, which are constructed from the topologies τ_j and ideals $\mathcal{I}$, as presented by M. Hosny (2020) [56] and later corrected by R. A. Hosny et al. (2022) [57]. This approach employs the closure ($cl_j^{\mathcal{I}}$) and interior ($int_j^{\mathcal{I}}$) operators of $\tau_j^{\mathcal{I}}$, rather than those of τ_j, to induce the class of $\mathcal{I}$-$\theta\beta_j$-open sets as an extension of the topologies $\tau_j^{\mathcal{I}}$.

However, the original work contained some typographical errors that require correction for scientific clarity. One such error occurred in Definition 4.1, where the index $\mathcal{I}$ was inadvertently omitted from the operators. This omission resulted in an inconsistent definition of the "$\mathcal{I}$-$\theta\beta_j$-open set."

In this article, we address these errors and present the necessary corrections. We clarify the identified error, provide the correct formulation, and reinforce the accurate concept initially presented in [55]. This clarification is supported by results and examples from the original research, ensuring consistency and correctness in the theoretical framework. Additionally, we present some essential and complementary results to the previous paper to offer a deeper understanding of its findings. We also introduce several new results and properties that both benefit rough set theory and expand its applications, supported by illustrative examples. Furthermore, we propose a new algorithm that facilitates the application of these methods in decision-making problems and in the development of additional real-life applications that require large-scale data. This study enriches the robustness and applicability of $\theta\beta$-ideal approximation spaces, contributing to advancements in fields such as decision-making, medicine, and data analysis.

The organization of this paper is as follows: Section **2** provides a review of foundational concepts, methodologies, and key results pertinent to this study, including a comparison with previous approaches. Section **3** addresses the typographical errors identified in [55] and presents the corresponding corrections. Section **4** introduces new results and examines the relationships between $\theta\beta$***-ideal rough sets*** and other related methods. Finally, Section **5** presents a mathematical algorithm intended to serve as a foundational framework for future applications.

2. Preliminaries

This section provides an overview of the fundamental concepts and their implications, serving as the foundation for the current study.

Definition 2.1 ([58])**.** A non-empty class $\mathcal{I}$ of subsets of a set X is called an *ideal* on X, if it satisfies the following conditions:
(i) if $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$ (hereditary),
(ii) if $A \in \mathcal{I}$ and $B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$ (finite additivity).

Definition 2.2 ([7, 9, 43, 44, 45, 46]). Let R be an arbitrary binary relation on a non-empty finite set X. For each $x \in X$, the j-neighborhoods of x (denoted by $N_j(x)$ for all $j \in \{r, \ell, \langle r\rangle, \langle \ell\rangle, i, u, \langle i\rangle, \langle u\rangle\}$) are defined as follows:
(i) *r-neighborhood*: $N_r(x) = \{y \in X : xRy\}$,
(ii) *ℓ-neighborhood*: $N_\ell(x) = \{y \in X : yRx\}$,
(iii) *$\langle r\rangle$-neighborhood*: $N_{\langle r\rangle}(x) = \bigcap_{x \in N_r(y)} N_r(y)$,
(iv) *$\langle \ell\rangle$-neighborhood*: $N_{\langle \ell\rangle}(x) = \bigcap_{x \in N_\ell(y)} N_\ell(y)$.
(v) *i-neighborhood*: $N_i(x) = N_r(x) \cap N_\ell(x)$,
(vi) *u-neighborhood*: $N_u(x) = N_r(x) \cup N_\ell(x)$,
(vii) *$\langle i\rangle$-neighborhood*: $N_{\langle i\rangle}(x) = N_{\langle r\rangle}(x) \cap N_{\langle \ell\rangle}(x)$,
(viii) *$\langle u\rangle$-neighborhood*: $N_{\langle u\rangle}(x) = N_{\langle r\rangle}(x) \cup N_{\langle \ell\rangle}(x)$.

Definition 2.3 ([44]). Let R be an arbitrary binary relation defined on a non-empty finite set X and let $\psi_j : X \to P(X)$ be a mapping that assigns a j-neighborhood to each $x \in X$ for all $j \in J = \{r, \ell, \langle r\rangle, \langle \ell\rangle, u, i, \langle u\rangle, \langle i\rangle\}$, where $P(X)$ denotes the power set of X. Then the triple (X, R, ψ_j) is referred to as a *j-neighborhood space* (briefly, j-**NS**).

Theorem 2.4 ([44]). *If (X, R, ψ_j) is a j-**NS**, then for each $j \in J$, the family*

$$\tau_j = \{A \subseteq X : \forall\, p \in A,\ N_j(p) \subseteq A\}$$

forms a topology on X.

Definition 2.5 ([44]). Let (X, R, ψ_j) be a j-**NS**. A subset $A \subseteq X$ is said to be a *j-open set*, if $A \in \tau_j$. The complement of a j-open set is called a *j-closed set*. The class of all j-closed sets is given by

$$\Gamma_j = \{F \subseteq X : F^c \in \tau_j\},$$

where F^c represents the complement of F in X.

Definition 2.6 ([44]). Let (X, R, ψ_j) be a j-**NS** and let $A \subseteq X$. For every $j \in J$, the *j-lower approximation*, *j-upper approximation*, *j-boundary region* and the *j-accuracy* of approximations of A are defined as follows:
(i) the *j-lower approximation*: $\underline{R}_j(A) = \bigcup\{G \in \tau_j : G \subseteq A\} = \mathrm{int}_j(A)$,
where $\mathrm{int}_j(A)$ represents the j-interior of A,
(ii) the *j-upper approximation*: $\overline{R}_j(A) = \bigcap\{F \in \Gamma_j : F \supseteq A\} = \mathrm{cl}_j(A)$,
where $\mathrm{cl}_j(A)$ represents the j-closure of A,
(iii) the *j-boundary region*: $B_j(A) = \overline{R}_j(A) - \underline{R}_j(A)$,
(vi) the *j-accuracy of approximation*: $\sigma_j(A) = \frac{|\underline{R}_j(A)|}{|\overline{R}_j(A)|}$,
where $|\overline{R}_j(A)| \neq 0$.

Definition 2.7 ([44]). Suppose that (X, R, ψ_j) is a j-**NS**, and let $A \subseteq X$. For each $j \in J$, the subset A is called a *j-exact set*, if

$$\underline{R}_j(A) = \overline{R}_j(A) = A.$$

Otherwise, A is called a *j-rough set*.

Theorem 2.8 ([56, 57]). *Let (X, R, ψ_j) be a j-**NS**, $A \subseteq X$, and let $\mathcal{I}$ be an ideal on X. For each $j \in J$, the class*

$$\tau_j^{\mathcal{I}} = \{A \subseteq X : \forall\, p \in A,\ N_j(p) \cap A^c \in \mathcal{I}\}$$

forms a topology on X. Here, A^c represents the complement of A.

Theorem 2.9 ([56, 57]). *Let (X, R, ψ_j) be a j-**NS** and let $\mathcal{I}$ be an ideal on X. Then*

$$\tau_j \subseteq \tau_j^{\mathcal{I}}, \quad \text{for each } j \in J.$$

Definition 2.10 ([56, 57]). Let (X, R, ψ_j) be a j-**NS** and let $\mathcal{I}$ be an ideal on X. A subset $A \subseteq X$ is called an $\mathcal{I}_j$*-open set*, if $A \in \tau_j^{\mathcal{I}}$. The complement of an $\mathcal{I}_j$-open set is called an $\mathcal{I}_j$*-closed set*. The family of all $\mathcal{I}_j$-closed sets is given by:

$$\Gamma_j^{\mathcal{I}} = \{F \subseteq X : F^c \in \tau_j^{\mathcal{I}}\}.$$

Definition 2.11 ([56, 57]). Let (X, R, ψ_j) be a j-**NS**, let $\mathcal{I}$ be an ideal on X and let $A \subseteq X$. For each $j \in J$, the $\mathcal{I}_j$*-lower approximation*, $\mathcal{I}_j$*-upper approximation*, $\mathcal{I}_j$*-boundary region* and $\mathcal{I}_j$*-accuracy* of the approximations of A are defined, respectively, as follows:

(i) the $\mathcal{I}_j$*-lower approximation*: $\underline{R}_j^{\mathcal{I}}(A) = \bigcup\{O \in \tau_j^{\mathcal{I}} : O \subseteq A\} = \mathrm{int}_j^{\mathcal{I}}(A)$, where $\mathrm{int}_j^{\mathcal{I}}(A)$ represents the $\mathcal{I}$-j-interior of A,

(ii) the $\mathcal{I}_j$*-upper approximation*: $\overline{R}_j^{\mathcal{I}}(A) = \bigcap\{F \in \Gamma_j^{\mathcal{I}} : F \supseteq A\} = \mathrm{cl}_j^{\mathcal{I}}(A)$, where $\mathrm{cl}_j^{\mathcal{I}}(A)$ represents the $\mathcal{I}$-j-closure of A,

(iii) the $\mathcal{I}_j$*-boundary region*: $B_j^{\mathcal{I}}(A) = \overline{R}_j^{\mathcal{I}}(A) - \underline{R}_j^{\mathcal{I}}(A)$,

(iv) the $\mathcal{I}_j$*-accuracy of the approximation*: $\sigma_j^{\mathcal{I}}(A) = \frac{|\underline{R}_j^{\mathcal{I}}(A)|}{|\overline{R}_j^{\mathcal{I}}(A)|}$, where $|\overline{R}_j^{\mathcal{I}}(A)| \neq 0$.

Nawar et al. successfully utilized the topological concept of the "θ-closure operator," originally introduced by Velicko in [59], within the framework of rough sets. They specifically employed the interior (int_j) and closure (cl_j) operators of the topologies generated by j-**NS** to define a novel class of near-open sets, referred to as "$\theta\beta_j$-open sets." This approach extended Pawlak's rough set theory and led to the development of generalized rough sets, termed "$\theta\beta_j$-rough sets," as formally defined below.

Definition 2.12 ([55]). Let (X, R, ψ_j) be a j-**NS** and let $A \subseteq X$. For each $j \in J$, the θ_j*-closure* of A is defined by

$$\mathrm{cl}_j^{\theta}(A) = \{x \in X : \text{for every } G \in \tau_j \text{ with } x \in G,\ A \cap \mathrm{cl}_j(G) \neq \varnothing\}.$$

Moreover, A is called θ_j*-closed*, if $A = \mathrm{cl}_j^{\theta}(A)$. The complement of a θ_j-closed set is said to be θ_j*-open*.

Note:

$$\mathrm{int}_j^{\theta}(A) = X \setminus \mathrm{cl}_j^{\theta}(X \setminus A).$$

Definition 2.13 ([55]). Let (X, R, ψ_j) be a j-**NS** and let $A \subseteq X$. A subset A is called a $\theta\beta_j$*-open set* if

$$A \subseteq \mathrm{cl}_j\Big[\mathrm{int}_j\big(\mathrm{cl}_j^{\theta}(A)\big)\Big] \quad \text{for each } j \in J.$$

A subset A is called a $\theta\beta_j$*-closed set*, if its complement is a $\theta\beta_j$-open set. The family of all $\theta\beta_j$-open sets and $\theta\beta_j$-closed sets is denoted by $\theta\beta_j O(X)$ and $\theta\beta_j C(X)$, respectively.

Definition 2.14 ([55])**.** Let (X, R, ψ_j) be a j-**NS** and let $A \subseteq X$. Then the $\theta\beta_j$*-lower approximation*, $\theta\beta_j$*-upper approximation*, $\theta\beta_j$*-boundary region* and $\theta\beta_j$*-accuracy* of the approximations of A are defined, respectively, as follows:

$$\underline{R}^{\theta\beta_j}(A) = \bigcup\{G \in \theta\beta_j O(X) : G \subseteq A\} \quad \text{(the } \theta\beta_j\text{-interior of } A),$$

$$\overline{R}^{\theta\beta_j}(A) = \bigcap\{F \in \theta\beta_j C(X) : F \supseteq A\} \quad \text{(the } \theta\beta_j\text{-closure of } A),$$

$$B^{\theta\beta_j}(A) = \overline{R}^{\theta\beta_j}(A) - \underline{R}^{\theta\beta_j}(A),$$

$$\sigma^{\theta\beta_j}(A) = \frac{|\underline{R}^{\theta\beta_j}(A)|}{|\overline{R}^{\theta\beta_j}(A)|}, \quad \text{where } |\overline{R}^{\theta\beta_j}(A)| \neq 0.$$

3. Some corrections to $\theta\beta$-ideal approximation spaces

In our original article [55], we introduced the concept of "$\theta\beta_j$-Ideal Approximation Spaces" by applying $\theta\beta_j$-open sets to the topologies $\tau_j^{\mathcal{I}}$. These generalized rough sets were constructed using the interior operator ($\mathrm{int}_j^{\mathcal{I}}$) and closure operator ($\mathrm{cl}_j^{\mathcal{I}}$) derived from $\tau_j^{\mathcal{I}}$, thereby extending Pawlak's rough set theory [1] and its subsequent enhancements, as discussed in [2, 7, 9, 15, 43, 44, 56, 57]. However, the original work contained typographical errors that require correction for scientific clarity.

One typographical error occurred in Definition 4.1, where the index $\mathcal{I}$ was inadvertently omitted from the operators. This omission led to an inconsistent definition of the "$\mathcal{I}$-$\theta\beta_j$-open set."

In this section, we address and rectify these issues. The main corrections are outlined below:

(1) *Definition 4.1* (on page 2488) contained an error due to the omission of the index $\mathcal{I}$ in the operators. The corrected form, now presented as Definition 3.1, includes the omitted index $\mathcal{I}$ to ensure consistency with the intended meaning and proper formulation.
(2) *Proposition 4.1* (on page 2488): The proof of this proposition essentially requires a lemma (see Lemma 3.2) to be properly illustrated. To ensure completeness, we present this lemma below as a preliminary (or complementary) result that establishes the relationship between the topologies generated by ideals, $\tau_j^{\mathcal{I}}$ (originally introduced by M. Hosny in [56] and later refined by R. A. Hosny et al. in [57]), and the class of $\mathcal{I}$-$\theta\beta_j$-open sets.
(3) *Example 5.1* (on pages 2492 and 2493) contained typographical errors, which have now been corrected in the current paper. These errors do not affect the analysis or the conclusions presented in the table.

Corrected Definition 4.1 of [55]:

Definition 3.1. Let (X, R, ψ_j) be a j-**NS**, and let $\mathcal{I}$ be an ideal on X. A subset $A \subseteq X$ is called an $\mathcal{I}$-$\theta\beta_j$*-open set*, if

$$A \subseteq \mathrm{cl}_j^{\mathcal{I}}\left[\mathrm{int}_j^{\mathcal{I}}\left(\mathrm{cl}_j^{*\theta}(A)\right)\right] \quad \text{for each } j \in J.$$

The complement of an $\mathcal{I}$-$\theta\beta_j$-open set is referred to as an $\mathcal{I}$-$\theta\beta_j$-*closed set*. The family of all $\mathcal{I}$-$\theta\beta_j$-open sets (resp. $\mathcal{I}$-$\theta\beta_j$-closed sets) is denoted by $\mathcal{I}$-$\theta\beta_j O(X)$ (resp. $\mathcal{I}$-$\theta\beta_j C(X)$).

Note:

$cl_j^{*\theta}(A) = A \cup A_j^{*\theta}$, where $A_j^{*\theta} = \Big\{x \in X : A \cap cl_j^{\mathcal{I}}(G) \notin \mathcal{I}, \forall G \in \tau_j^{\mathcal{I}} \text{ with } x \in G\Big\}$.

Essential Lemma for Proposition 4.1 of [55]: We provide the following lemma to establish the relationship between the topology $\tau_j^{\mathcal{I}}$ and the class of $\mathcal{I}$-$\theta\beta_j$-open sets:

Lemma 3.2. *Let (X, R, ψ_j) be a j-**NS**, let $\mathcal{I}$ be an ideal on X and let $\tau_j^{\mathcal{I}}$ be the topology on X generated by $\mathcal{I}$. Then*

$$\tau_j^{\mathcal{I}} \subseteq \mathcal{I}\text{-}\theta\beta_r O(X).$$

Proof. Let $A \in \tau_j^{\mathcal{I}}$. By definition, we have

$$A = \mathrm{int}_j^{\mathcal{I}}(A). \tag{3.1}$$

From the definition of $cl_j^{*\theta}(A)$, it follows that

$$A \subseteq cl_j^{*\theta}(A),$$

which implies

$$\mathrm{int}_j^{\mathcal{I}}(A) \subseteq \mathrm{int}_j^{\mathcal{I}}\Big(cl_j^{*\theta}(A)\Big).$$

Accordingly, by (3.1), we obtain

$$A \subseteq \mathrm{int}_j^{\mathcal{I}}\Big(cl_j^{*\theta}(A)\Big),$$

which further implies

$$\mathrm{cl}_j^{\mathcal{I}}(A) \subseteq \mathrm{cl}_j^{\mathcal{I}}\Big[\mathrm{int}_j^{\mathcal{I}}\Big(cl_j^{*\theta}(A)\Big)\Big].$$

Since, as noted in [56, 57], we have $A \subseteq \mathrm{cl}_j^{\mathcal{I}}(A)$ for every $A \in \tau_j^{\mathcal{I}}$, it follows that

$$A \subseteq \mathrm{cl}_j^{\mathcal{I}}\Big[\mathrm{int}_j^{\mathcal{I}}\Big(cl_j^{*\theta}(A)\Big)\Big].$$

Thus we conclude that $A \in \mathcal{I}$-$\theta\beta_r O(X)$. □

Remark 3.3. As demonstrated in the following example (Example 3.4), the converse of Lemma 3.2 does not hold in general.

Example 3.4. According to Example 4.1 of [55], consider

$$X = \{a, b, c, d, e\},$$

the binary relation

$$R = \{(a, a),\ (a, e),\ (b, a),\ (b, c),\ (b, d),\ (b, e),\ (c, c),\ (c, d),\ (d, c),\ (d, d),\ (e, e)\},$$

and the ideal

$$\mathcal{I} = \{\varnothing, \{a\}, \{c\}, \{a, c\}\}.$$

Then the right neighborhoods are:

$$N_r(a) = \{a, e\}, \quad N_r(b) = \{a, c, d, e\}, \quad N_r(c) = N_r(d) = \{c, d\}, \quad N_r(e) = \{e\}.$$

Accordingly, the family $\mathcal{I}$-$\theta\beta_r O(X)$ is given by:
$\mathcal{I}$-$\theta\beta_r O(X) = \{X, \varnothing, \{b\}, \{d\}, \{e\}, \{a,b\}, \{a,e\}, \{b,c\}, \{b,d\}, \{b,e\}, \{c,d\}, \{d,e\}, \{a,b,c\}, \{a,b,d\}, \{a,b,e\}, \{a,d,e\}, \{b,c,d\}, \{b,c,e\}, \{b,d,e\}, \{c,d,e\}, \{a,b,c,d\}, \{a,b,c,e\}, \{a,b,d,e\}, \{a,c,d,e\}, \{b,c,d,e\}\}$.
However, $\tau_r^{\mathcal{I}} = \{X, \varnothing, \{d\}, \{e\}, \{a,e\}, \{c,d\}, \{d,e\}, \{a,d,e\}, \{b,d,e\}, \{c,d,e\}, \{a,b,d,e\}, \{a,c,d,e\}, \{b,c,d,e\}\}$.
It is clear that there are subsets in $\mathcal{I}$-$\theta\beta_r O(X)$ that are not members of $\tau_r^{\mathcal{I}}$. For instance, $\{a,b,c,d\} \in \mathcal{I}$-$\theta\beta_r O(X)$, but $\{a,b,c,d\} \notin \tau_r^{\mathcal{I}}$.

Corrections of some typos in Example 5.1 of [55]:

Example 3.5. All right neighborhoods for each element of

$$H = \{c_1, c_2, c_3, c_4, c_5\}$$

are given by:

$$c_1R = \bigcap_{s=1}^{5} c_1R_s = \{c_1, c_4\}, \quad c_2R = \bigcap_{s=1}^{5} c_2R_s = \{c_2, c_5\},$$

$$c_3R = \bigcap_{s=1}^{5} c_3R_s = \{c_2, c_3, c_4, c_5\}, \quad c_4R = \bigcap_{s=1}^{5} c_4R_s = \{c_4\}, \quad c_5R = \bigcap_{s=1}^{5} c_5R_s = \{c_5\}.$$

The topology generated by these right neighborhoods is given by:
$\tau_r = \{H, \varnothing, \{c_4\}, \{c_5\}, \{c_1, c_4\}, \{c_2, c_5\}, \{c_4, c_5\}, \{c_1, c_4, c_5\}, \{c_2, c_4, c_5\}, \{c_1, c_2, c_4, c_5\}, \{c_2, c_3, c_4, c_5\}\}$.

According to the Chemistry expert, if $\mathcal{I} = \{\varnothing, \{c_1\}, \{c_4\}, \{c_1, c_4\}\}$ is the selected ideal, then the topology generated by this ideal is given by:

$\tau_r^{\mathcal{I}} = \{H, \varnothing, \{c_1\}, \{c_4\}, \{c_5\}, \{c_1, c_4\}, \{c_1, c_5\}, \{c_2, c_5\}, \{c_4, c_5\}, \{c_1, c_2, c_5\}, \{c_1, c_4, c_5\}, \{c_2, c_3, c_5\}, \{c_2, c_4, c_5\}, \{c_1, c_2, c_3, c_5\}, \{c_1, c_2, c_4, c_5\}, \{c_2, c_3, c_4, c_5\}\}$.

Then we have

$$\mathcal{I}\text{-}\theta\beta_r O(H) = P(H),$$

where $P(H)$ is the power set of H.

Thus the typos in Table 5 (of Example 5.1) on pages 2492 and 2493 have been corrected in Table 2, which presents a comparison between our approach and the previous ones in [44] and [56]:

4. New Results on $\theta\beta$-ideal approximation spaces

This section introduces new results and properties of $\theta\beta$-Ideal Approximation Spaces, as initially proposed in [55]. Additionally, we examine their theoretical framework and illustrate these concepts with examples, offering a more comprehensive understanding of their structure and significance.

TABLE 1. Comparison of the Boundary Region and Accuracy Measures Using the Techniques of Abd El-Monsef et al. [44], Hosny [56], and the Proposed Method (Definition 4.2 [55])

$A \subseteq H$	Abd El-Monsef method		M. Hosny method		The current method	
	$B_r(A)$	$\sigma_r(A)$	$B_r^{\mathcal{I}}(A)$	$\sigma_r^{\mathcal{I}}(A)$	$B_r^{\mathcal{I}-\theta\beta}(A)$	$\sigma_r^{\mathcal{I}-\theta\beta}(A)$
$\{c_1\}$	$\{c_1\}$	0	$\varnothing$	1	$\varnothing$	1
$\{c_2\}$	$\{c_2, c_3\}$	0	$\{c_2, c_3\}$	0	$\varnothing$	1
$\{c_3\}$	$\{c_3\}$	0	$\{c_3\}$	0	$\varnothing$	1
$\{c_4\}$	$\{c_1, c_3\}$	1/3	$\varnothing$	1	$\varnothing$	1
$\{c_5\}$	$\{c_2, c_3\}$	1/3	$\{c_2, c_3\}$	1/3	$\varnothing$	1
$\{c_1, c_2\}$	$\{c_1, c_2, c_3\}$	0	$\{c_2, c_3\}$	1/3	$\varnothing$	1
$\{c_1, c_3\}$	$\{c_1, c_3\}$	0	$\{c_3\}$	1/2	$\varnothing$	1
$\{c_1, c_4\}$	$\{c_3\}$	2/3	$\varnothing$	1	$\varnothing$	1
$\{c_1, c_5\}$	$\{c_1, c_2, c_3\}$	1/4	$\{c_2, c_3\}$	1/2	$\varnothing$	1
$\{c_2, c_3\}$	$\{c_2, c_3\}$	0	$\{c_2, c_3\}$	0	$\varnothing$	1
$\{c_2, c_4\}$	$\{c_1, c_2, c_3\}$	1/4	$\{c_2, c_3\}$	1/3	$\varnothing$	1
$\{c_2, c_5\}$	$\{c_3\}$	2/3	$\{c_3\}$	2/3	$\varnothing$	1
$\{c_3, c_4\}$	$\{c_1, c_3\}$	1/3	$\{c_3\}$	1/2	$\varnothing$	1
$\{c_3, c_5\}$	$\{c_2, c_3\}$	1/3	$\{c_2, c_3\}$	1/3	$\varnothing$	1
$\{c_4, c_5\}$	$\{c_1, c_2, c_3\}$	2/5	$\{c_2, c_3\}$	1/2	$\varnothing$	1
$\{c_1, c_2, c_3\}$	$\{c_1, c_2, c_3\}$	0	$\{c_2, c_3\}$	1/3	$\varnothing$	1
$\{c_1, c_2, c_4\}$	$\{c_2, c_3\}$	1/2	$\{c_2, c_3\}$	1/2	$\varnothing$	1
$\{c_1, c_2, c_5\}$	$\{c_1, c_3\}$	1/2	$\{c_3\}$	3/4	$\varnothing$	1
$\{c_1, c_3, c_4\}$	$\{c_3\}$	2/3	$\{c_3\}$	2/3	$\varnothing$	1
$\{c_1, c_3, c_5\}$	$\{c_1, c_2, c_3\}$	1/4	$\{c_2, c_3\}$	1/2	$\varnothing$	1
$\{c_1, c_4, c_5\}$	$\{c_2, c_3\}$	3/5	$\{c_2, c_3\}$	3/5	$\varnothing$	1
$\{c_2, c_3, c_4\}$	$\{c_1, c_2, c_3\}$	1/4	$\{c_2, c_3\}$	1/3	$\varnothing$	1
$\{c_2, c_3, c_5\}$	$\{c_3\}$	2/3	$\varnothing$	1	$\varnothing$	1
$\{c_2, c_4, c_5\}$	$\{c_1, c_3\}$	3/5	$\{c_3\}$	3/4	$\varnothing$	1
$\{c_3, c_4, c_5\}$	$\{c_1, c_2, c_3\}$	2/5	$\{c_2, c_3\}$	1/2	$\varnothing$	1
$\{c_1, c_2, c_3, c_4\}$	$\{c_2, c_3\}$	1/2	$\{c_2, c_3\}$	1/2	$\varnothing$	1
$\{c_1, c_2, c_3, c_5\}$	$\{c_1, c_3\}$	1/2	$\varnothing$	1	$\varnothing$	1
$\{c_1, c_2, c_4, c_5\}$	$\{c_3\}$	4/5	$\{c_2, c_3\}$	4/5	$\varnothing$	1
$\{c_1, c_3, c_4, c_5\}$	$\{c_2, c_3\}$	3/5	$\{c_2, c_3\}$	3/5	$\varnothing$	1
$\{c_2, c_3, c_4, c_5\}$	$\{c_1\}$	4/5	$\varnothing$	1	$\varnothing$	1
H	$\varnothing$	1	$\varnothing$	1	$\varnothing$	1

Definitions and Preliminary concepts.
Nawar et al. [55] introduced rough approximations based on the class of $\mathcal{I}$-$\theta\beta_j$-open sets. The corrected formulation of these approximations, as refined in Definition 3.1, is presented in the revised definition below.

Definition 4.1. Let (X, R, ψ_j) be a j-**NS**, and let $\mathcal{I}$ be an ideal on X. For each $A \subseteq X$ and for all $j \in J$, the following concepts are defined:

(i) the $\mathcal{I}$-$\theta\beta_j$-*lower approximation*:

$$\underline{R}_j^{\mathcal{I}-\theta\beta}(A) = \bigcup\{G \in \mathcal{I}\text{-}\theta\beta_j O(X) : G \subseteq A\} = \mathrm{int}_j^{\mathcal{I}-\theta\beta}(A),$$

where $\mathrm{int}_j^{\mathcal{I}-\theta\beta}(A)$ represents the $\mathcal{I}$-$\theta\beta_j$-interior of A,

(ii) the $\mathcal{I}$-$\theta\beta_j$-*upper approximation*:

$$\overline{R}_j^{\mathcal{I}-\theta\beta}(A) = \bigcap\{F \in \mathcal{I}\text{-}\theta\beta_j C(X) : F \supseteq A\} = \mathrm{cl}_j^{\mathcal{I}-\theta\beta}(A),$$

where $\mathrm{cl}_j^{\mathcal{I}-\theta\beta}(A)$ represents the $\mathcal{I}$-$\theta\beta_j$-closure of A,

(iii) the $\mathcal{I}$-$\theta\beta_j$-*boundary region*:

$$B_j^{\mathcal{I}-\theta\beta}(A) = \overline{R}_j^{\mathcal{I}-\theta\beta}(A) - \underline{R}_j^{\mathcal{I}-\theta\beta}(A).$$

(iv) the $\mathcal{I}$-$\theta\beta_j$-*accuracy of the approximations*:

$$\sigma_j^{\mathcal{I}-\theta\beta}(A) = \frac{\left|\underline{R}_j^{\mathcal{I}-\theta\beta}(A)\right|}{\left|\overline{R}_j^{\mathcal{I}-\theta\beta}(A)\right|}, \quad \text{where} \quad \left|\overline{R}_j^{\mathcal{I}-\theta\beta}(A)\right| \neq 0.$$

Reformulation of Results from [55].
We now reformulate certain results from [55] to clarify relationships between the approximations introduced in Definition 4.1 and those in Definitions 2.10 and 2.11.

Theorem 4.2. *Let (X, R, ψ_j) be a j-**NS**, and let $A \subseteq X$. If $\mathcal{I}$ is an ideal on X, then the following statements hold:*

(1) $\underline{R}_j(A) \subseteq \underline{R}_j^{\mathcal{I}}(A) \subseteq \underline{R}_j^{\mathcal{I}-\theta\beta}(A)$,
(2) $\overline{R}_j^{\mathcal{I}-\theta\beta}(A) \subseteq \overline{R}_j^{\mathcal{I}}(A) \subseteq \overline{R}_j(A)$.

Proof. We will prove the first part, and the second follows similarly.
From Theorem 2.9, we have:

$$\underline{R}_j(A) = \bigcup\{G \in \tau_j : G \subseteq A\} \subseteq \bigcup\{G \in \tau_j^{\mathcal{I}} : G \subseteq A\} = \underline{R}_j^{\mathcal{I}}(A).$$

Additionally, from Lemma 3.2, we obtain:

$$\underline{R}_j^{\mathcal{I}}(A) = \bigcup\{G \in \tau_j^{\mathcal{I}} : G \subseteq A\} \subseteq \bigcup\{G \in \mathcal{I}\text{-}\theta\beta_j O(X) : G \subseteq A\} = \underline{R}_j^{\mathcal{I}-\theta\beta}(A).$$

□

Corollary 4.3. *Let (X, R, ψ_j) be a j-**NS**, and let $A \subseteq X$. If $\mathcal{I}$ is an ideal on X, then the following statements hold:*

(1) $B_j^{\mathcal{I}-\theta\beta}(A) \subseteq B_j^{\mathcal{I}}(A) \subseteq B_j(A)$,
(2) $\sigma_j(A) \leq \sigma_j^{\mathcal{I}}(A) \leq \sigma_j^{\mathcal{I}-\theta\beta}(A)$.

Corollary 4.4. *Let (X, R, ψ_j) be a j-**NS**, $\mathcal{I}$ be an ideal on X, and $A \subseteq X$. Then the following statements hold:*

(1) *each j-exact subset in X is $\mathcal{I}$-$\theta\beta_j$-exact,*
(2) *each $\mathcal{I}$-j-exact subset in X is $\mathcal{I}$-$\theta\beta_j$-exact,*
(3) *each $\mathcal{I}$-$\theta\beta_j$-rough subset in X is j-rough,*
(4) *each $\mathcal{I}$-$\theta\beta_j$-rough subset in X is $\mathcal{I}$-j-rough.*

Remark 4.5. (1) Based on Theorem 4.2 and its corollaries, the $\mathcal{I}$-$\theta\beta_j$-rough set models serve as generalizations of the methods introduced by Abd El-Monsef et al. [44] and M. Hosny [56].

(2) As demonstrated in Example 3.5 (presented in Table 1), it is evident that the converse of these results does not generally hold.

(3) Furthermore, as shown in Example 4.1 of [55], the class of $\theta\beta_j$-open sets (respectively, $\mathcal{I}$-$\theta\beta_j$-open sets) constitutes a supra-topological space under a general binary relation.

(4) Additionally, the following results explore specific cases of relations. Notably, they demonstrate that the topologies generated under the condition of a symmetric relation are quasi-discrete spaces (where every open set is closed). Consequently, the class of $\theta\beta_j$-open sets (respectively, $\mathcal{I}$-$\theta\beta_j$-open sets) forms a discrete topological space.

Lemma 4.6 ([52, 60])**.** *Let (X, R, ψ_j) be a j-**NS**. If R is a symmetric relation, then for each $x \in X$,*

(1) $y \in N_r(x) \Leftrightarrow x \in N_r(y)$.
(2) $N_r(x) = N_\ell(x) = N_i(x) = N_u(x)$.
(3) $N_{\langle r\rangle}(x) = N_{\langle \ell\rangle}(x) = N_{\langle i\rangle}(x) = N_{\langle u\rangle}(x)$.

Theorem 4.7. *Let (X, R, ψ_j) be a j-**NS**. If R is a symmetric relation, then for all $j \in \{r, \ell, u, i\}$, the topology τ_j is a quasi-discrete space.*

Proof. We will prove the theorem for the case of $j = r$. By Lemma 4.6, the result extends to the other cases of j.

First, suppose $A \in \tau_r$. Then by Theorem 2.4,

$$N_r(p) \subseteq A \quad \text{for every } p \in A. \tag{4.1}$$

Now, let $y \in A^c$. We consider the following cases:

Case 1: $N_r(y) \subseteq A^c$. In this case, it follows directly that $A^c \in \tau_r$.

Case 2: $N_r(y) \subseteq A$. By 4.1, this implies $y \in A$, which contradicts the assumption $y \in A^c$.

Case 3: $N_r(y) \cap A \neq \varnothing$. This means there exists $w \in X$ such that $w \in A$ and $w \in N_r(y)$. Since R is symmetric, by Lemma 4.6, we have $y \in N_r(w)$. This leads to a contradiction, as it implies that $w \in A$ and $N_r(w) \cap A^c \neq \varnothing$, which contradicts the assumption in 4.1.

From the analysis of **cases (1)-(3)**, we conclude that $N_r(y) \subseteq A^c$ for every $y \in A^c$, which implies $A^c \in \tau_r$. Thus τ_r is a quasi-discrete space. □

Remark 4.8. For any j-**NS** (X, R, ψ_j), if R is a symmetric relation, then for all $j \in \{\langle r\rangle, \langle \ell\rangle, \langle i\rangle, \langle u\rangle\}$, the topology τ_j does not necessarily have to be a quasi-discrete space, as demonstrated in the following example.

Example 4.9. Consider the symmetric relation $R = \{(a, a), (a, b), (b, a), (c, c), (d, a), (a, d)\}$ on the set $X = \{a, b, c, d\}$. Thus we have:

So $\tau_r = \{X, \varnothing, \{c\}, \{a, b, d\}\}$ and $\tau_{\langle r\rangle} = \{X, \varnothing, \{a\}, \{c\}, \{a, c\}, \{a, b, d\}\}$.

Theorem 4.10. *Let (X, R, ψ_j) be a j-**NS**. If R is a symmetric relation, then for all $j \in \{r, \ell, u, i\}$, the class $\theta\beta_jO(X)$ (resp. $\mathcal{I}$-$\theta\beta_jO(X)$) forms a discrete topological space.*

TABLE 2. r-neighborhoods and $\langle r\rangle$- neighborhoods

x	$N_r(x)$	$N_{\langle r\rangle}(x)$
a	$\{a,b,d\}$	$\{a\}$
b	$\{a\}$	$\{a,b,d\}$
c	$\{c\}$	$\{c\}$
d	$\{a\}$	$\{a,b,d\}$

Proof. We will prove the result for $\theta\beta_j O(X)$; the proof for $\mathcal{I}$-$\theta\beta_j O(X)$ follows similarly. From Theorem 4.7, we have $\text{int}_j(A) = \text{cl}_j(A)$ for every $A \subseteq X$. Thus every subset $A \subseteq X$ satisfies $A \in \theta\beta_j O(X)$. This implies that $\theta\beta_j O(X)$ is the power set of X. Consequently, $\theta\beta_j O(X)$ forms a discrete topological space. □

Proposition 4.11. *Let (X, R, ψ_j) be a j-**NS**. If R is a symmetric relation, then for all $j \in \{r, \ell, u, i\}$ and all $A \subseteq X$,*

(1) $\underline{R}_j^{\theta\beta}(A) = \overline{R}_j^{\theta\beta}(A) = A$,
(2) $\underline{R}_j^{\mathcal{I}-\theta\beta}(A) = \overline{R}_j^{\mathcal{I}-\theta\beta}(A) = A$.

Proof. The result directly follows from Theorem 4.10. □

Corollary 4.12. *Let (X, R, ψ_j) be a j-**NS**. If R is a symmetric relation, then for all $j \in \{r, \ell, u, i\}$ and all $A \subseteq X$,*

(1) $B_j^{\theta\beta}(A) = B_j^{\mathcal{I}-\theta\beta}(A) = \varnothing$,
(2) $\sigma_j^{\theta\beta}(A) = \sigma_j^{\mathcal{I}-\theta\beta}(A) = 1$.

Remark 4.13. Proposition 4.11 and Corollary 4.12 highlight a crucial aspect of our method under a symmetric relation, demonstrating that it surpasses earlier approaches formulated using topological structures and their extensions. Moreover, the proposed method is more effective than prior techniques that directly utilize neighborhood systems or those combining neighborhood systems and ideal structures. The following example illustrates this distinction.

Example 4.14. Consider Example 4.9. We compute the approximations of certain subsets using $\theta\beta_j$-rough sets and $\mathcal{I}$-$\theta\beta_j$-rough sets as follows:
Suppose $\mathcal{I} = \{\varnothing, \{a\}, \{c\}, \{a, c\}\}$. Then the results are:

■ **Method of Abd El-Monsef et al. (2014)**:
The generated topologies are:

$$\tau_r = \{X, \varnothing, \{c\}, \{a, b, d\}\} \quad \text{and} \quad \tau_{\langle r\rangle} = \{X, \varnothing, \{a\}, \{c\}, \{a, c\}, \{a, b, d\}\}.$$

Accordingly, the classes of all closed sets are given by:

$$\Gamma_r = \{X, \varnothing, \{c\}, \{a, b, d\}\} \quad \text{and} \quad \Gamma_{\langle r\rangle} = \{X, \varnothing, \{c\}, \{b, d\}, \{a, b, d\}, \{b, c, d\}\}.$$

Thus it is clear that the only exact subsets are $\{c\}$ and $\{a, b, d\}$ in both cases of $j = r$ and $j = \langle r\rangle$.

■ **Method of M. Hosny method (2020)**:
For $j = r$, the generated topology is:

$$\tau_r^{\mathcal{I}} = \{X, \varnothing, \{b\}, \{c\}, \{d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, d\}, \{b, c, d\}\}.$$

Thu, the class of all closed sets is:

$$\Gamma_r^{\mathcal{I}} = \{X, \varnothing, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}\}.$$

So it is clear that the only exact subsets are $\{c\}$ and $\{a, b, d\}$ in the case of $j = r$.
For $j = \langle r\rangle$, the generated topology is:

$$\tau_{\langle r\rangle}^{\mathcal{I}} = \{X, \varnothing, \{a\}, \{c\}, \{a, c\}, \{b, d\}, \{a, b, d\}, \{b, c, d\}\}.$$

Hence, the class of all closed sets is:

$$\Gamma_{\langle r\rangle}^{\mathcal{I}} = \{X, \varnothing, \{a\}, \{c\}, \{a, c\}, \{b, d\}, \{a, b, d\}, \{b, c, d\}\}.$$

Therefore, it is clear that the only exact subsets are $\{a\}$, $\{c\}$, $\{a, b, d\}$, and $\{b, c, d\}$ in the case of $j = \langle r\rangle$.

■ **Method of $\theta\beta_j$ and $\mathcal{I}$-$\theta\beta_j$-rough sets**:

It is evident that for every $A \subseteq X$, A is an exact subset according to the $\theta\beta_j$- and $\mathcal{I}$-$\theta\beta_j$-approaches. This demonstrates that the proposed methods ($\theta\beta_j$- and $\mathcal{I}$-$\theta\beta_j$-rough sets) provide greater accuracy than the earlier methods proposed by Abd El-Monsef et al. and M. Hosny.

To illustrate this, consider the subset $M = \{a, b, c\}$. Then, the accuracy measures of M using the methods of Abd El-Monsef et al. and M. Hosny are as follows: $\sigma_r(M) = \frac{1}{4}$, $\sigma_{\langle r\rangle}(M) = \frac{1}{2}$, $\sigma_r^{\mathcal{I}}(M) = \frac{2}{3}$, and $\sigma_{\langle r\rangle}^{\mathcal{I}}(M) = \frac{1}{2}$.

In contrast, the accuracy measures of M using the $\theta\beta_j$- and $\mathcal{I}$-$\theta\beta_j$-rough set methods are:

$$\sigma_{\langle r\rangle}^{\theta\beta}(M) = \sigma_{\langle r\rangle}^{\mathcal{I}-\theta\beta}(M) = 1.$$

5. Algorithmic framework for $\theta\beta$-ideal approximation spaces

Lastly, we introduce Algorithm 1 outlining the use of our proposed techniques to assist in decision-making tasks. This algorithm is easily implementable using simple programming languages like MATLAB, facilitating high-precision medical diagnosis.

Algorithm 1: Determining exactness and roughness using $\theta\beta$-ideal approximation spaces

(1) **Input Data:** Construct the dataset and define the binary relation R on the set X.
(2) **Compute Similarities:** Compute the similarity degree

$$\mathcal{S}(x, y) = \frac{\sum_{i=1}^{n} [a_i(x) = a_i(y)]}{n},$$

where n is the number of condition attributes. Construct a similarity table.

(3) **Establish Binary Relation:** Define
$$(x,y) \in R \Leftrightarrow \mathcal{S}(x,y) \geq \epsilon,$$
where ϵ is the similarity threshold. Generate a similarity matrix.
(4) **Neighborhoods Calculation:** Compute all j-neighborhoods $N_j(x)$ for each $x \in X$.
(5) **Ideal Set Selection:** Choose an ideal set $\mathcal{I}$ based on expert knowledge.
(6) **Computation of Topologies:** Compute $\tau_j^{\mathcal{I}}$ and $\Gamma_j^{\mathcal{I}}$:
$$\tau_j^{\mathcal{I}} = \{A \subseteq X : \forall x \in A,\ N_j(x) \cap A^c \in \mathcal{I}\},\ and\ \Gamma_j^{\mathcal{I}} = \{F \subseteq X : F^c \in \tau_j^{\mathcal{I}}\}.$$
(7) **Computation of $\mathcal{I}$-$\theta\beta_j$-Classes:** Compute $\mathcal{I}$-$\theta\beta_jO(X)$ and $\mathcal{I}$-$\theta\beta_jC(X)$ using Definition 3.1.
(8) **Rough Set Identification:** For $A \subseteq X$, compute
$$\underline{R}_j^{\mathcal{I}-\theta\beta}(A) = \bigcup\{G \in \mathcal{I}\text{-}\theta\beta_jO(X) : G \subseteq A\}.$$
If $\underline{R}_j^{\mathcal{I}-\theta\beta}(A) = \varnothing$, conclude that A is a rough set.
(9) **$\mathcal{I}$-$\theta\beta_j$-Accuracy Determination:** Compute the upper approximation $\overline{R}_j^{\mathcal{I}-\theta\beta}(A)$ and accuracy:
$$\sigma_j^{\mathcal{I}-\theta\beta}(A) = \frac{\left|\underline{R}_j^{\mathcal{I}-\theta\beta}(A)\right|}{\left|\overline{R}_j^{\mathcal{I}-\theta\beta}(A)\right|}, \quad \text{if} \quad \left|\overline{R}_j^{\mathcal{I}-\theta\beta}(A)\right| \neq 0.$$
If $\sigma_j^{\mathcal{I}-\theta\beta}(A) = 1$, classify A as exact; otherwise, it is rough.
(10) **Decision Outcome:** Analyze decision outcomes and validate against benchmarks.
(11) **Adjust Parameters (if necessary):** Modify R based on results and repeat.
(12) **Output Results:** Present final results and insights.

Algorithm 1 analysis: The following analysis discusses the effectiveness, efficiency, and scalability of Algorithm 1.

Effectiveness: The core objective of Algorithm 1 is to distinguish between exact and rough sets by computing binary relations and $\mathcal{I}$-$\theta\beta_j$-approximations of j-neighborhoods. The algorithm achieves high classification accuracy, provided that precise data and rigorous definitions are used. Its iterative approach, based on well-established neighborhood computations and $\mathcal{I}$-$\theta\beta_j$-accuracy measures (denoted as $\sigma_j^{\mathcal{I}-\theta\beta}$), ensures that each decision is grounded in solid theoretical principles.

Efficiency: The efficiency of Algorithm 1 is dependent on the dataset size. As the volume of data increases, the complexity associated with binary relation and neighborhood computations grows. By implementing iterative recalculations in optimized programming environments (such as Python or R), the algorithm effectively manages computational demands. Furthermore, the use of advanced data structures and memorization (caching) techniques can minimize redundant operations, thereby reducing the overall runtime.

Scalability: Although the algorithm's iterative nature provides a systematic framework for classification, scalability may become a concern for extremely large datasets. To mitigate this, one can employ strategies such as parallel processing and data reduction techniques. Modern programming languages, with robust libraries for handling big data, enable the algorithm to be adapted for real-time or large-scale applications, ensuring its practical viability across diverse domains, including medical diagnosis, economic modeling, and beyond.

6. Conclusion

This study advances both the theoretical foundation and practical applications of $\theta\beta$-ideal approximation spaces by refining key definitions, introducing new results, and proposing a comprehensive algorithm for real-world implementation. The presented framework not only rectifies typographical inaccuracies from our previous work but also extends and reinforces its theoretical underpinnings, ensuring greater consistency and scientific rigor. Moreover, our approach generalizes earlier methodologies, including those developed by Abd El-Monsef et al. [44] and M. Hosny [56], offering enhanced accuracy and broader applicability. By integrating theoretical innovations with practical examples, we have demonstrated the effectiveness of the refined $\theta\beta$-ideal rough set models compared to existing techniques. The proposed mathematical algorithm provides a robust computational framework adaptable to various programming environments, facilitating its application in fields such as medicine, chemistry, and economics. This contribution highlights the practical significance of our study in decision-making contexts.
Looking ahead, future research will focus on further generalizations and the exploration of novel applications in emerging fields, such as generalized multi-granulation [61], soft nodec spaces [62], and N-Bipolar Soft Expert Sets and their Applications [38]. These directions aim to address the growing demand for precise data analysis and advanced decision-support systems. The advancements presented in this paper serve as a foundation for ongoing research in topological and rough set theory, offering valuable insights for both theoretical progress and practical implementation.

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